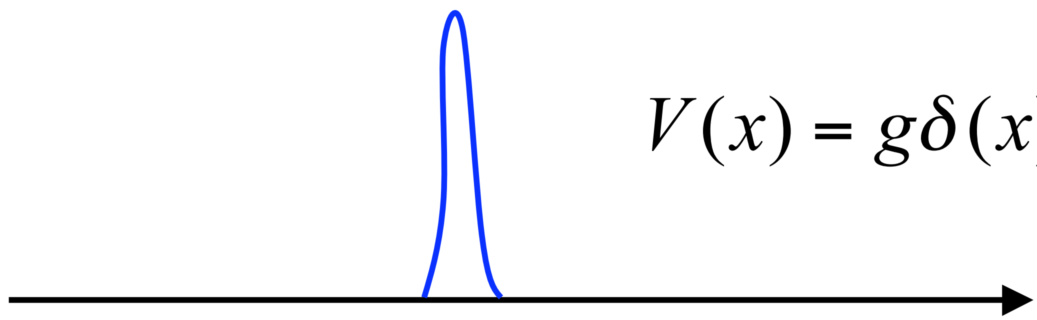
Lecture 18:
Delta-function Scattering

Phy851 Fall 2009

Delta-Function Scatterer

- Any very narrow barrier can be approximated by a delta function:



$$V(x) = g\delta(x)$$

- The coefficient g is then the area under $V(x)$:

  $g = h \times w$

  $= \text{Energy} \times \text{Length}$

- Conditions for validity of delta-function approximation:
  - Incoming wave characterized by k, which gives a length-scale: $\lambda = 2\pi/k$

    Thus we surely must require: $w \ll \lambda$ → $k w \ll 1$

  - But the delta-function must have another length scale associated with it (from V_0)
    - *Based on units only, we find a second length scale, let's call it `a':*

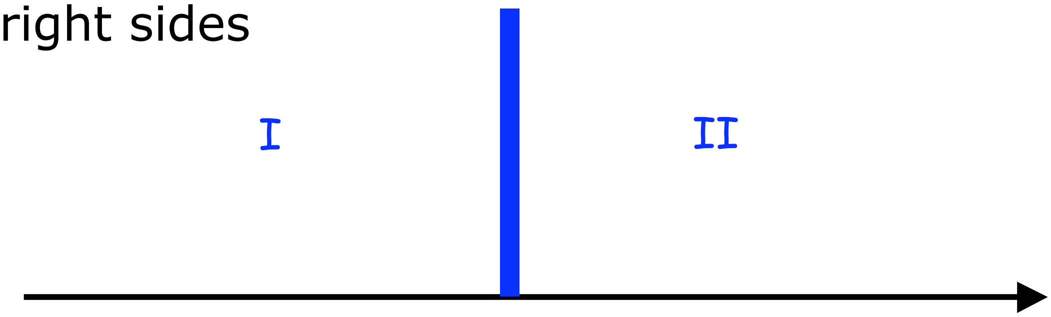
$$g = \frac{\hbar^2}{ma} = \frac{\hbar^2}{ma^2} a \qquad a = \frac{\hbar^2}{mg}$$

Do we also Need: $k a \ll 1$?

In the limit $w \to 0$, scattering is governed by the scattering length

Delta-Function Scatterer

- Scattering by the delta-function will be handled by applying boundary conditions to connect the wavefunctions on the left and right sides



$$\psi_1(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_2(x) = Ce^{ikx} + De^{-ikx}$$

- RECALL: a delta-function in the potential means that ψ_(x) is discontinuous
  - But $\psi(x)$ remains continuous

- PRIMARY GOAL: Determine the proper boundary conditions for _ and _´ at the location of a delta function scatterer
  - Be able to solve `plug and chug' problems

- Secondary Goal: find M_δ for the delta potential:

$$\begin{pmatrix} C \\ D \end{pmatrix} = M_\delta \begin{pmatrix} A \\ B \end{pmatrix}$$

Delta-function Boundary Condition

- All boundary conditions are derived from Schrödinger's Equation:

$$E\psi(x) = -\frac{\hbar^2}{2m}\psi''(x) + g\delta(x)\psi(x)$$

- For the delta-potential, the trick is to integrate both sides from $-\varepsilon$ to $+\varepsilon$
  - Then take limit as $\varepsilon \to 0$

$$E\int_{-\varepsilon}^{\varepsilon} dx\,\psi(x) = -\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon} dx\,\psi''(x) + g\int_{-\varepsilon}^{\varepsilon} dx\,\delta(x)\psi(x)$$

$$E\psi(0)2\varepsilon = -\frac{\hbar^2}{2m}\big(\psi'(\varepsilon) - \psi'(-\varepsilon)\big) + g\psi(0)$$

  - Take $\varepsilon \to 0$: $\psi'(\varepsilon) \to \psi_2'(0)$ $\psi'(-\varepsilon) \to \psi_1'(0)$

$$0 = -\frac{\hbar^2}{2m}\big(\psi_2'(0) - \psi_1'(0)\big) + g\psi(0)$$

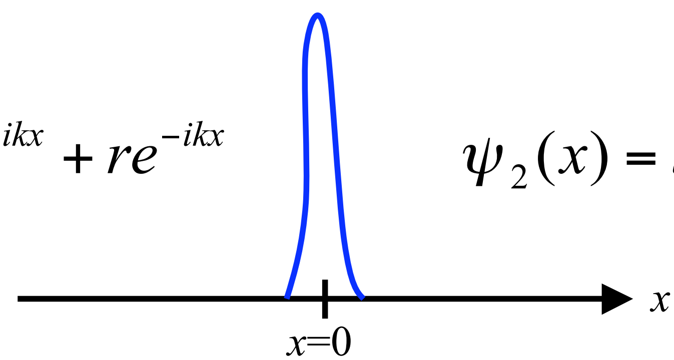
$$\psi_2'(0) = \psi_1'(0) + \frac{2mg}{\hbar^2}\psi(0)$$

Example

- Lets solve the delta-potential scattering problem via `plug and chug’ method:

  – Q: Let $V(x)$ = g _(x). For a single incident wave with momentum k, what are the reflection and transmission amplitudes and Probabilities?

$$\psi_1(x) = e^{ikx} + re^{-ikx} \qquad \psi_2(x) = te^{ikx}$$



$$\int_{-\epsilon}^{\epsilon} dx\, E\psi = -\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\psi''\,dx + g\int_{-\epsilon}^{\epsilon}\delta(x)\,dx\,\psi$$

$$E\psi(0)2\epsilon = -\frac{\hbar^2}{2m}\left(\psi'(\epsilon) - \psi'(-\epsilon)\right) + g\psi(0)$$

$$\psi_2'(0) = \psi_1'(0) + \frac{2mg}{\hbar^2}\psi(0)$$

Solution:

$$\psi_1(x) = e^{ikx} + r e^{-ikx}$$

$$\psi_2(x) = t e^{ikx}$$

$$\psi_1(0) = 1 + r \qquad \psi_2(0) = t$$

$$\psi_1'(0) = ik(1-r) \qquad \psi_2'(0) = ikt$$

$$\psi_1(0) = \psi_2(0) \rightarrow \quad 1 + r = t$$

$$\psi_2'(0) = \psi_1'(0) + \frac{2mg}{\hbar^2}\psi(0) \rightarrow \quad ikt = ik(1-r) + \frac{2mg}{\hbar^2}(1+r)$$

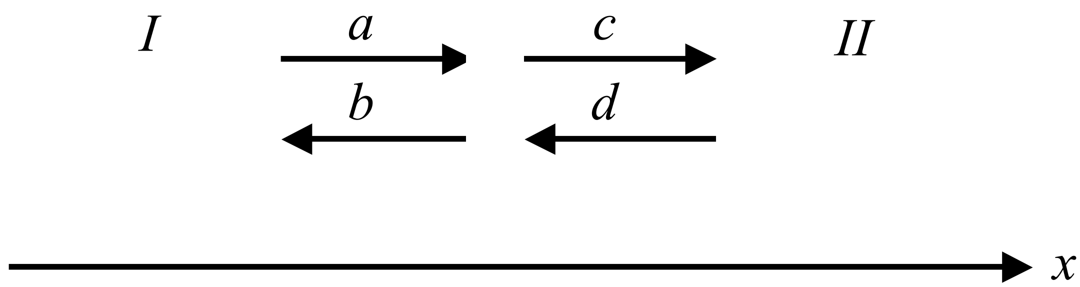
$$1 + r = 1 - r - i\frac{2mg}{\hbar^2 k}(1+r)$$

$$\left(2 + \frac{2img}{\hbar^2 k}\right) r = -i\frac{2mg}{\hbar^2 k}$$

$$r = \frac{-i\frac{mg}{\hbar^2 k}}{1 + i\frac{mg}{\hbar^2 k}} = \frac{-1}{1 - i\frac{\hbar^2 k}{mg}}$$

$$r = \frac{-1}{1 - ika} \qquad R = \frac{1}{1 + (ka)^2} \quad T = \frac{(ka)^2}{1 + (ka)^2}$$

Transfer Matrix for Delta function



$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$
$$\psi_{II}(x) = Ce^{ikx} + De^{-ikx}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = M_\delta \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\psi_2(0) = \psi_1(0)$$
$$\psi_2'(0) = \psi_1'(0) + \frac{2}{a}\psi(0)$$

$$a = \frac{\hbar^2}{mg}$$

$$C + D = A + B$$ b.c. 1

$$ik(C - D) = ik(A - B) + \frac{2}{a}(A + B)$$

$$C - D = A - B - i\frac{2}{ka}(A + B)$$ b.c. 2

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 - i\frac{2}{ka} & -1 - i\frac{2}{ka} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

Continued

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 - i\dfrac{2}{ka} & -1 - i\dfrac{2}{ka} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$M_\delta(ka) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 1 - i\dfrac{2}{ka} & -1 - i\dfrac{2}{ka} \end{pmatrix}$$

$$M_\delta(ka) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 - i\dfrac{2}{ka} & -1 - i\dfrac{2}{ka} \end{pmatrix}$$

$$M_\delta(ka) = \begin{pmatrix} 1 - \dfrac{i}{ka} & -\dfrac{i}{ka} \\ \dfrac{i}{ka} & 1 + \dfrac{i}{ka} \end{pmatrix}$$

$$V_\delta(x) = g\delta(x)$$

$$a = \frac{\hbar^2}{mg}$$

Summary of Transfer Matrix Results:

- Basic Elements:

$$M_{free}(kL) = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix}$$

$$M_{step}(k_2, k_1) = \frac{1}{2k_2}\begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix}$$

$$M_\delta\left(ka\right) = \frac{1}{ika}\begin{pmatrix} ika + 1 & 1 \\ -1 & ika - 1 \end{pmatrix}$$

- For *n* regions (n-1 boundaries):

$$M = M^{[n,n-1]} M_f^{[n-1]} M^{[n-1,n-2]} \ldots M^{[3,2]} M_f^{[2]} M^{[2,1]}$$

$$R = \left|\frac{M_{12}}{M_{22}}\right|^2 \qquad T = 1 - R$$