RESEARCH STATEMENT

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1. Summary and Goals

I am an algebraic geometer working in the area of tropical geometry which is a method for associating objects called *tropical varieties* to algebraic varieties. These tropical varieties, which are polyhedral complexes, are much simpler than the associated algebraic varieties but often capture a surprising amount of information about them. They can be studied combinatorially and can be thought of as a combinatorial approximation to the Berkovich analytification of the original variety. My goal is to understand this process of *tropicalization* by

- Studying which polyhedral complexes arise from tropicalization by developing specific combinatorial obstructions;
- Understanding how the monodromy of a family of varieties is reflected in its tropicalization;
- Finding a dictionary between the algebraic geometry of varieties with smooth tropicalizations and the combinatorics of their tropicalizations;
- Developing enumerative geometry and linear systems on tropical surfaces;
- Generalizing the Baker-Norine theory of linear systems on graphs by studying coherent cohomology on general tropical varieties;
- Systematically developing tropical Schubert calculus by using ideas from Bruhat-Tits theory.

2. Introduction to Tropical Geometry

Discussion of my work begins the next section. This section provides tropical background.

The main theme of tropical geometry is transforming questions about algebraic varieties into more combinatorial questions about polyhedral complexes. One begins with an algebraic variety X, the common zero set of a system of polynomial equations in an algebraic torus $(\mathbb{K}^*)^n$ defined over a valued field $\mathbb{K}$. By the method of *tropicalization*, one can define a *tropical variety*, $\mathrm{Trop}(X)$, which is a polyhedral complex, as a combinatorial shadow of X. The combinatorics of $\mathrm{Trop}(X)$ reflects the algebraic geometry of X.

Tropical geometry originally arose from considering algebraic geometry over the tropical semiring $(\mathbb{T}, \oplus, \otimes)$ whose underlying set $\mathbb{T}$ is the real numbers with operations given by

$$a \oplus b = \min(a, b), \quad a \otimes b = a + b.$$

One can then find tropical analogues of classical mathematics and define tropical polynomials, tropical hypersurfaces, and tropical varieties. These objects do not look like their classical counterparts and instead are polyhedral complexes of differing combinatorial types. A number of people have developed tropical geometry by defining the appropriate analogues of notions from algebraic geometry and by showing that analogous theorems hold. Other results in this direction show that enumerative questions have the same answers tropically and classically. A spectacular early result of Mikhalkin [Mi03] established that the number of plane curves of degree d with genus g passing through through $3d - 1 + g$ general points could be computed using tropical geometry. With collaborators, he found tropical analogues of theorems about algebraic curves and further developed the enumerative geometry of curves [Mi06, MZ08]. Gathmann, H. Markwig, and collaborators have transferred much of Gromov-Witten theory over to tropical geometry. [Ga06, Mar07]

An approach to tropical geometry, which originated in an idea of Kapranov, is to define a tropical variety as a shadow of an algebraic variety [EKL06]. Let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of formal Puiseux series, that is, the field of Laurent series where exponents may be fractions but with bounded denominator (one may also use other algebraically closed fields with valuations). This field has a valuation $v : \mathbb{K}^* \to \mathbb{Q} \subset \mathbb{R}$. One considers a subvariety of an algebraic torus, $X \subset (\mathbb{K}^*)^n$, and defines its tropical variety as $\mathrm{Trop}(X) = \overline{v(X)}$, the closure of the image of X under the product of valuation maps $v : (\mathbb{K}^*)^n \to \mathbb{R}^n$. This process of constructing $\mathrm{Trop}(X)$ from X is called *tropicalization*. It is known from the work of Bergman and Bieri-Groves that $\mathrm{Trop}(X)$ is a polyhedral complex of dimension equal to that of X. The affine span of each polyhedral cell is rational affine space. It was shown by Speyer [Sp05] that the resulting complex has natural weights and satisfies a certain balancing condition. If X is defined over $\mathbb{C}$, then $\mathrm{Trop}(X)$ is a fan. We call such balanced weighted rational polyhedral complexes *tropical varieties*. Those that arise from a classical variety by tropicalization are called *tropicalizations*. This approach to tropical geometry is tied to the theory of Gröbner bases in combinatorial algebraic geometry.

$\mathrm{Trop}(X)$ is very closely related to the dual complex of a degeneration of X over a DVR when X is defined over a valued field $\mathbb{K}$. In the case where X is defined over $\mathbb{C}$, by a result of Tevelev [Te07], $\mathrm{Trop}(X)$ is related to the dual complex of a compactification of X. Therefore, $\mathrm{Trop}(X)$ captures the combinatorics of stratifications (in the $\mathbb{C}$ case) and of degenerations (in the $\mathbb{K}$ case). In the complex case, tropical geometry is a combinatorial theory of subvarieties of a toric variety stratified by intersections with toric strata. In that sense, tropical geometry is an enlargement of the theory of toric varieties. In the valued field case, understanding $\mathrm{Trop}(X)$ comes down to understanding the combinatorics of a degeneration of X. There is a certain tension between algebraic geometry and combinatorics in tropical geometry: if the combinatorics of $\mathrm{Trop}(X)$ are simple, the algebraic geometry of the components is likely to be complicated and rich; if the components of the degeneration are simple, the combinatorics of $\mathrm{Trop}(X)$ are rich and capture the geometry of X.

The tropicalization of X reflects many of the properties of X. $\mathrm{Trop}(X)$ contains a lot of information about the intersection theory of X which is why enumerative tropical geometry has been successful. Moreover, $\mathrm{Trop}(X)$ captures properties of the monodromy of X considered as a family of a punctured disc. It is a subtle question to determine whether a polyhedral complex is the tropicalization of an algebraic variety. In a certain sense tropicalizations are very special among polyhedral complexes and should have additional structures to reflect these properties.

Tropical varieties are novel because they encode degenerations with much more complicated combinatorics than previously studied. For example, Cools, Draisma, Payne, and Robeva [CFPR] were able to give a new proof of the Brill-Noether theorem by degenerating a higher genus curve into a union of rational curves with complicated dual graph and then bounding the dimension of a linear system on the smooth curve using the specialization lemma of Baker [Ba08]. This is orthogonal to the approach using limit linear series where the dual graphs are all trees [EH86]. Moreover, tropical geometry gives an explicit way of approaching Berkovich spaces which are a certain type of analytic spaces: it is a theorem of Payne [P09] that the Berkovich analytification X^{an} of an affine variety X is homeomorphic to a certain inverse limit of tropicalizations $\varprojlim \mathrm{Trop}(X, \iota)$. In a certain sense, this result says that one can refine the tropicalization of X and keep track of additional combinatorial data so that it contains any piece of information about the algebraic geometry of X but at the expense of complicated combinatorics. In this sense, any given tropicalization can be viewed as an approximation to the Berkovich space and properties of the Berkovich space can be seen (or at least approximated) by the tropicalization.

There are many applications of tropical geometry. The work of Hacking, Keel, and Tevelev [HKT09] constructs compactifications of moduli spaces of del Pezzo surfaces using tropical geometry. Tillmann [Ti05] has used tropical geometry to study ideal points in the space of hyperbolic structures on 3-manifolds. Recent work by Gubler [Gu07a, Gu07b] and Rabinoff [R09] have applied tropical techniques to answering questions about the arithmetic of abelian varieties. Tropical intersection theory [AR10] generalizes Newton polytope techniques in number theory [R10] and is likely to have many applications there in the future. Tropical geometry is also used in the approaches to mirror symmetry taken by Kontsevich-Soibelman [KS06] and Gross-Siebert [GS06, GS07]. Both approaches involve an integral affine structure on a polyhedral complex. In one approach, it comes from a rigid analytic Calabi-Yau manifold. In the other, it comes from a degeneration of a Calabi-Yau manifold. Tropical geometry is related to numerical homotopy methods [HS95] in scientific computing. It has also been applied to phylogenetics in mathematical biology [PS08] and to integrable systems [IT08]. In Fall 2009, Tropical Geometry was the subject of a semester-long MSRI program of which I was a postdoctoral member.

3. Results

3.1. **Tropical Realization Spaces.** Given a polyhedral complex Σ, it is natural to ask if Σ is the tropicalization of an algebraic variety. This is called the *tropical lifting* or *tropical realization problem*. I explain specific obstructions to a complex being a tropicalization below but here I discuss a modular approach to lifting. One wishes to consider the set of all varieties $V \subset (\mathbb{K}^*)^n$ with $\mathrm{Trop}(V) = \Sigma$ and show that they form a nice moduli space called the *tropical realization space*. I have two papers (one with Sam Payne) in this direction.

Payne and I have studied realizability questions for weighted fans $(\mathcal{F}, m)$ in $\mathbb{R}^n$ [KP09]. Here one considers varieties $X \subset (\mathbb{C}^*)^n$ with $\mathrm{Trop}(X) = (\mathcal{F}, m)$. This case is already very rich - Mikhalkin has given an example of a two-dimensional tropical fan that can only be the tropicalization of a variety defined over a field of characteristic 2. We have proven that the moduli functor for varieties with fixed tropical variety is representable by an algebraic space (and if $\mathcal{F}$ is quasiprojective, a scheme of finite type). There is a distillation of non-realizability results called *Murphy's Law* [Va06] which shows that in a certain sense, a moduli space can be arbitrarily pathological. We have proven that realization spaces (and therefore lifting phenomena) obey Murphy's Law. This shows that realization questions can be arbitrarily complicated.

In [Ka10a], I studied the case of realization spaces over discretely valued fields. For (Σ, m), a weighted polyhedral complex in $\mathbb{R}^n$, in this case, the realization space is the parameter space consisting of all varieties $X \subset (\mathbb{K}^*)^n$ satisfying $\mathrm{Trop}(X) = (\Sigma, m)$. Because this constrains the central fiber of X over a DVR, one must work with analytic conditions rather than algebraic ones. I have shown that the (suitably defined) realization space parameterizing varieties with tropicalization (Σ, m) is a rigid analytic space. Because of the use of rigid analytic geometry, this is the best that can be hoped for. I have used these results to establish a *density Tropical Lefschetz Principle* which shows that a property established by analytic techniques (say Hodge theory or by considering Hausdorff limits of amoebas) is true for formal families over $\mathbb{C}\{\{t\}\}$. I like this result because it uses the combinatorics of Chow polytopes together with rigid analytic geometry.

3.2. **The Topology of Tropical Varieties and Hodge Theory.** Let $f : Y^\circ \to \mathbb{D}^*$ be an algebraic family of subvarieties of $(\mathbb{C}^*)^n$ defined over the punctured disc. Suppose that Y° is schön (a technical but natural smoothness condition). After a possible base-change, we may complete Y° to a smooth, connected, (complex) $(d+1)$-dimensional manifold Y with a proper map $f : Y \to \mathbb{D}$ to the unit disc $\mathbb{D}$ which is smooth over the punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. This completion depends on the choice of a polyhedral complex Σ. Let Y_x denote a generic fiber of this family. The fundamental group of the punctured disc, $\mathbb{Z}$, gives a monodromy action on the cohomology $H^*(Y_x)$ by parallel transport.

3.2.1. *Relating* $\mathrm{Trop}(Y^\circ)$ *to the monodromy of* Y°. My paper with David Helm [HKa08] studies the monodromy filtration on the cohomology of a generic fiber and relates it to the geometry of $\mathrm{Trop}(Y^\circ)$. The results in this paper relate the algebraic geometry of Y to the tropical geometry of $\mathrm{Trop}(Y^\circ)$. After possible base-change, the monodromy operator T becomes unipotent and induces a filtration on the cohomology $H^r(Y_x)$ of a generic fiber. The lowest part of the filtration which is of degree $-r$ is described by a complex related to $\mathrm{Trop}(Y^\circ)$: there is a balanced integral polyhedral complex called the *parameterizing complex* Γ_{Y° together with a surjective piecewise-linear map $\Gamma_{Y^\circ} \to \mathrm{Trop}(Y^\circ)$ such that $H^r(\Gamma^\circ_Y, \mathbb{Q}) \cong H^r(Y_x, \mathbb{Q})_{-r}$. This puts strong constraints on the topology of Γ_{Y° and therefore on $\mathrm{Trop}(Y^\circ)$. This theorem also implies a vanishing theorem for the cohomology of tropicalization of complete intersections which is a non-constant coefficient analogue of a result of Hacking. [Ha08]

This work also allows one to understand more explicitly the highest power of the monodromy operator acting on the middle-dimensional cohomology of Y_x. In fact, the action of $N = \mathrm{Log}(T)$ is encoded in the geometry of Γ_{Y°: the action $N^d : H^d(Y_x, \mathbb{Q}) \to H^d(Y_x, \mathbb{Q})$ is described by the map $H_d(\Gamma_{Y^\circ}, \mathbb{Q}) \to H^d(\Gamma_{Y^\circ}, \mathbb{Q})$ induced from the "volume pairing" on the parameterizing complex Γ°_Y which takes a pair of n-dimensional cycles to the (oriented) lattice volume of their intersection. This result naturally generalizes my work with H. Markwig and T. Markwig [KMM08, KMM09] on the tropical j-invariants of elliptic curves.

3.2.2. *Tropical Geometry and Hodge Theory.* My work with Alan Stapledon on the tropical motivic nearby fiber [KaS10] extracts data about a certain mixed Hodge structure on Y_x from the data of the tropical variety and initial degenerations. There is a limit mixed Hodge structure [Ste75] on the cohomology on a generic fiber of f which is denoted by Y_∞. By work of Bittner [Bi05], there is a natural invariant of the family f called the motivic nearby fiber. This is a sort of master invariant for monodromy. It is given as a class ψ_f in the Grothendieck group of varieties over $\mathbb{C}$, $K_0(\mathrm{Var}_{\mathbb{C}})$ and specializes to the limit Hodge-Deligne polynomial, $E(Y_\infty; u, v)$. We define an invariant, $\psi_{(Y^\circ,\Sigma)}$ *the tropical motivic nearby fiber* which, when Y° is schön and the recession fan of Σ is smooth, is equal to the motivic nearby fiber. It is constructed from the data of Γ_{Y° and the initial degenerations of Y°. This allows us to describe the limit Hodge-Deligne polynomial of a generic fiber and consequently the Euler characteristic in terms of combinatorics of the tropicalization and the algberaic geometry of initial degenerations.

There are certain cases where the $\psi_{(Y^\circ,\Sigma)}$ is determined entirely by combinatorics. One is the case of families of hypersurfaces of $(\mathbb{C}^*)^n$. The geometry of the initial degenerations is described combinatorially by the work of Danilov and Khovanskiĭ [DK86]. By encompassing these results, we are able to give a formula for the limit Hodge-Deligne polynomial of the family of hypersurfaces purely in terms of tropical data. The other case, that of varieties with smooth tropicalizations, is described below.

In future work with Patrick Brosnan, Stapledon and I are going to study the relative weight filtration on the cohomology of the open variety Y°. This will get us a more geometric understanding of the tropical motivic nearby fiber. We also have very simple degeneration-theoretic proofs of some results of the Khovanskiĭ school on the geometry of hypersurfaces in algebraic tori.

3.3. **Varieties with smooth tropicalization.** In a particular sense, varieties with smooth tropicalization are the simplest varieties after toric varieties and using them, one can pursue the project of relating algebraic geometric and combinatorial properties. Smooth tropical varieties are polyhedral complexes locally modeled on the matroid fans of Ardila and Klivans [AK06]. A variety with smooth tropicalization can be degenerated into a union of linear spaces which can be understood entirely in terms of the combinatorics of its tropicalization. With Stapledon, I have proven that if a variety $Y^\circ \subset (\mathbb{K}^*)^n$ has smooth tropicalization then Y° is automatically schön. This implies that Y° is smooth and has a well-behaved degeneration over the disc. Therefore, any property of Y° that survives degeneration should be reflected in the combinatorics. Because the open strata of the central fiber are hyperplane arrangement complements, their motivic classes are determined by their matroids as in the work of Orlik-Solomon [OS80]. This yields a combinatorial formula for the tropical motivic nearby fiber, Hodge-Deligne polynomial and Euler characteristic in terms of its local matroids [KaS10].

I suspect that much more is true. On the algebraic geometric side, by computing the differentials of the Steenbrink spectral sequence, we should be able to give a combinatorial formula for the Hodge numbers $h^{p,q}(H^m(Y_\infty))$. On the combinatorial side, there should be algebraic geometric interpretations of matroid invariants like *cd*-indices and Tutte polynomials. I plan to investigate the correspondence between matroid and algebraic geometric invariants and find a dictionary.

3.4. **Tropical Realizability.** A natural question is the following: *Let Σ be a polyhedral complex. Is Σ the tropicalization of a variety?* This question is combinatorial in nature but is very subtle in that it is not immediate to distinguish tropicalizations among the polyhedral complexes. There are variations on this question:

Problem 1. *(Lifting Problem) Given a balanced weighted rational polyhedral complex $\Sigma \subset \mathbb{R}^n$, when does there exist a subvariety $V \subset (\mathbb{K}^*)^n$ with $\mathrm{Trop}(V) = \Sigma$.*

Problem 2. *(Relative Lifting Problem) Let $W \subset (\mathbb{K}^*)^n$ be a subvariety. Given a balanced weighted rational polyhedral complex $\Sigma \subset \mathrm{Trop}(W)$, when does there exist a subvariety $V \subset W$ with $\mathrm{Trop}(V) = \Sigma$.*

These questions have significant applications outside of tropical geometry. Gibney and Maclagan [GM10, Mac] have found a connection between relative lifting and Fulton's conjecture about the effective cone of $\overline{\mathcal{M}}_{0,n}$. Bogart, Brugallé, and Cotterill have a program to prove Clemens's conjecture that all rational curves on a quintic threefold are rigid by showing that the rational tropical curves on the tropicalization of the threefold are rigid. Unfortunately, there are often non-rigid tropical curves which are not tropicalizations. A class of such examples in the tropicalization of a surface were constructed by Vigeland [Vi10]. Below, I will discuss a combinatorial criterion that allows one to rule out such curves.

The most interesting case of lifting problems to consider are curves. Questions of when a weighted balanced rational graph Γ in $\mathbb{R}^n$ is the tropicalization of a curve have been addressed by Speyer [Sp05], Brugallé-Mikhlkin, Nishinou [N09], and Tyomkin [Ty10]. In general, such a curve lifts when it moves in a family of the expected dimension. When it does not, necessary and sufficient conditions are well-understood in genus 1 by work of Speyer and partially understood in higher genera by work of Nishinou [N09] and forthcoming work of Brugallé-Mikhalkin and Tyomkin.

3.4.1. *Absolute Lifting Problem for Curves.* Inspired by the work of Nishinou-Siebert [NS06], I have found a necessary condition for non-regular space curves to lift that replicates all other known necessary conditions and gives new obstructions for higher genus curves in cases where the other conditions do not apply. I have approached lifting problems for curves from the point of view of deformation theory of log stable maps of curves to toric varieties. There is a particular heuristic for tropical lifting where one first uses the tropicalization as a blueprint for constructing the central fiber of a semistable degeneration of the prospective lift, then one uses deformation theory to extend central fiber to a legitimate lift. The obstruction for lifting curves in space is in the deformation theoretic step. By relating deformation theory and combinatorics, I've proven the necessity of a condition for graphs Σ to arise as tropicalizations. This condition is phrased in the language of linear systems on graphs as developed by Baker-Norine [BN07]. My condition involves the existence of tropical 1-forms φ_m that arise as combinatorial shadows of log 1-forms, $\omega_m = f^*\frac{dz^m}{z^m}$ where z^m is a character of $(\mathbb{K}^*)^n$. I study the necessity of lifting the tropical 1-forms φ_m to classical 1-forms. From this, I am able to reproduce all the known necessary conditions for curve lifting. [Sp05, N09] I also have new conditions in cases where those do not apply. In some sense, this work unearths an unexpected combinatorial nature to deformation theory. The 1-forms φ_m provide an additional structure on embedded tropical curves that I expect to fit into the log geometry framework of Gross-Siebert [GS07]. I hope to soon explore the sufficiency of my lifting criteria. This is likely to require a combinatorial understanding of the Kuranishi map. Some results in that direction have been achieved by Nishinou [N09]. Surprisingly (to me), this method of understanding a curve by looking at the restriction to the central fiber of a related 1-form is very similar to Coleman's method of effective Chabauty in the bad reduction case as explored by Lorenzini-Tucker [LT02] and McCallum-Poonen [MP].

In the near future, I hope to extend this work to higher dimensions. While it is unrealistic to expect there to be sufficient conditions to guarantee that a polyhedral complex lifts, the necessary conditions may put interesting additional combinatorial structures on tropicalizations that encode the algebraic geometry of the original variety.

3.4.2. *Relative Lifting Problem for Curves in Hypersurfaces.* Another variation of the lifting problem that I've investigated is the *relative lifting problem* for curves in hypersurfaces: Let $V(f) \subset (\mathbb{K}^*)^n$ be a hypersurface. Let $\Gamma \subset \mathrm{Trop}(V)$ be a balanced weighted rational graph. Does Γ lift to a curve on V? The example of Vigeland falls into this situation. In Figure 1 is pictured a tropical line in a plane and a tropical line in some cells of Vigeland's surface. In both cases, the line and the ambient surface lift. However, the line can lift to a subvariety of the plane but never to a subvariety of Vigeland's surface.

In work with Bogart [BKa10], I have found a necessary condition for such a curve to lift by studying factorizations of polynomials in the spirit of [Stu96]. It applies to unimodular hypersurfaces which are cut out by polynomials $f \in \mathbb{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$ whose Newton subdivisions consist of unimodular simplexes. The condition limits the local geometry near vertices of the tropical curve. The obstruction prevents one from constructing a particular broken

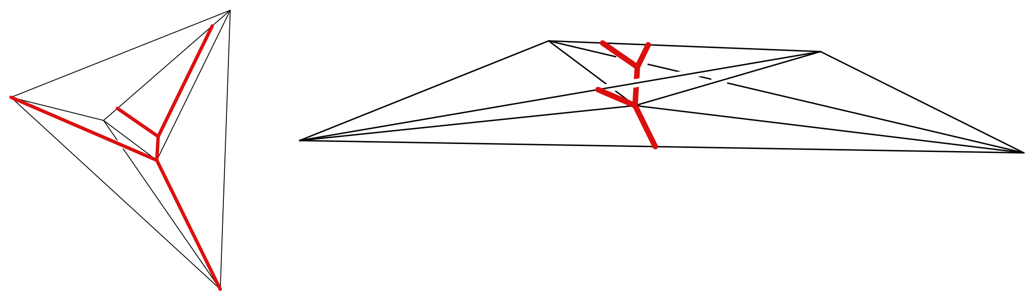

Figure 1. Tropical line in a tropical plane and in a piece of Vigeland's surface

curve in the central fiber of a degeneration of the hypersurface. This theorem shows that all of Vigeland's lines are spurious resolving some issues that had remained open and mysterious in tropical geometry.

3.5. **Other work.** I do not write much about my work on tropical intersection theory [Ka09b, Ka09c] or my work with Payne on piecewise polynomials [KP08] except that I expect to use them in future work on enumerative geometry. They give an idea which enumerative properties are visible to tropical geometry. My paper on *Tropical Invariants of the Secondary Fan* [Ka09a] can be interpreted as intersection theory on a moduli space of degenerations of toric varieties. This is likely to be of use when studying curve counts relative to a non-smooth divisor in the spirit of Li's approach to relative Gromov-Witten theory [L01]. The intersection theory on this moduli space can yield relations among enumerative invariants as in the paper [Ka07] based on my thesis research.

4. Ongoing Projects

4.1. **Counting Tropical Curves on K3 and Abelian Surfaces.** Mikhalkin's results on curve counting are about counting curves on projective toric surfaces. It is natural to extend these results to more general surfaces. The enumerative geometry of classical curves in Abelian and K3 surfaces is well-understood so one has many test cases with which to sharpen the tropical theory.

Helm and I have approached the case of Abelian surfaces [HKa]. One should be able to adapt the approach of Nishinou-Siebert [NS06] which involves considering degenerations of toric varieties into broken toric varieties, counting broken curves in broken varieties, and then verifying that they smooth. Abelian varieties have degenerations to broken toric varieties due to Mumford [Mu72]. We have verified that the classical and tropical curve counts agree for an infinite family of examples.

The K3 surface case is much richer. The degeneration theory of K3 surfaces is well-understood [FM83]. The most interesting degenerations are those of Type III which have dual complex a sphere. However, the degenerations lack some nice properties of the Abelian surface case. In particular, some components of the central fiber are not toric varieties. This introduces singularities in the integral affine structure. There is machinery developed by Gross-Siebert [GS07] to handle this case. In fact, the enumeration will involve tropical curves in the non-singular part of the dual complex and scattering terms that keep track of components of curves in the non-toric components of the central fiber.

4.2. **Tropical Linear Systems.**

4.2.1. *Extending the Baker-Norine theory of linear systems to polyhedral surfaces.* The Baker-Norine theory of linear systems [BN07] on graphs is a way of estimating the dimension of a linear system on a curve by considering the combinatorics of a degeneration. Given a curve C defined over a discretely valued field, one considers a regular semistable model $\mathcal{C}$ with central fiber $\mathcal{C}_0$ and dual graph Γ. Any divisor on D on C induces a divisor $\rho(D)$ on Γ. Baker and Norine give a combinatorial definition of the rank of $\rho(D)$ on Γ which is an upper bound for the dimension of any linear system on C containing D by Baker's specialization lemma [Ba08]. One can study the linear system on Γ intrinsically and develop Riemann-Roch and Abel-Jacobi theory [BN07]. Such work is closely related to the chip-firing games studied by the graph theory community [Bi99, BLS91]. The space of tropical linear systems has been studied combinatorially by Haase, Musiker, and Yu [HMY09].

I would like to extend this theory to surfaces. Here, one considers divisors which are balanced graphs on abstract tropical surfaces which are two-dimensional polyhedral complexes equipped with an integral affine structure. There is a natural notion of linear equivalence and rank of divisors. In this case, there is a significant problem in that the abstract definition of a tropical surface is missing, but I have proven Baker's specialization lemma [Ka11] in the case where the surface is the parameterizing complex of the tropicalization of a classical surface. The proof uses the intersection-theoretic technology that I developed in [Ka09c]. This shows that my definitions are correct and gives some hints as to the theory of abstract tropical surfaces. The combinatorial theory of such tropical linear systems is bound to be at least as rich as the theory of linear systems on curves. Haase, Musiker, Stapledon, and I plan to study this.

4.2.2. *Cohomology theory of coherent sheaves on complexes.* While the Baker-Norine theory of linear systems on graphs is parallel to the classical theory on curves and has theorems analogous to Riemann-Roch and Abel-Jacobi [BN07], the arguments make use of involved combinatorics. In fact, one does not yet have a useful notion of higher cohomology.

I would like to understand the cohomology of coherent sheaves on integral affine complexes. In the case where the complex is the dual complex of a normal crossings degeneration of a variety X, I have an approach in terms of Cech cohomology on the analytification X^{an}. One takes a covering of the analytification X^{an} by the inverse images of strata of the central fiber. By Payne's result relating the Berkovich analytification of X to the inverse limit of tropicalizations [P09], it should be theoretically possible to recover the sheaf cohomology of X. I hope to investigate this point of view to define a combinatorial analogue of the higher cohomology groups. The situation is considerably simpler in the case of graphs, and I have a candidate definition of h^1 and the first steps of a proof of tropical Riemann-Roch. I hope to proceed by quite general arguments to a combinatorial understanding of Riemann-Roch, Serre duality, and eventually Grothendieck-Riemann-Roch.

4.3. **Tropical Schubert Calculus.** Schubert calculus is intersection theory on flag manifolds and has resisted translation into tropical geometry. The approach I outline here which is joint work with Maria Angelica Cueto, currently a graduate student at UC-Berkeley – while speculative at this stage – makes contact between Schubert calculus and Bruhat-Tits theory. Schubert calculus deals with intersections of Schubert varieties which are natural cycles in a flag manifold and which depend on a choice of flag. In theory, one may take tropicalizations of Schubert varieties lying in tropical general position and intersect them, but in practice this is very difficult because ensuring tropical general position puts a combinatorial description beyond practical reach. Instead, we plan to understand Schubert varieties as double Bruhat cells in a partial flag variety, G/P over the discrete valuation ring $\mathcal{O} = \mathbb{C}[[t]]$.

Tropicalization does not respect group actions so particular care must be taken to establish a theory in which one can talk of orbits of a Borel subgroup. In other words, one should consider a homogeneous version of tropicalization adapted to the problem. The Bruhat-Tits building of $G = \mathrm{Sl}_n(\mathbb{C}((t)))$ which keeps track of the specialization properties of an element together with its conjugates is the natural object to use for such a theory.

4.4. **Curves in Hyperkahler Manifolds.** I have recently begun a project with Andrew Neitzke, a mathematical physicist at University of Texas. We are interested in coming up with a tropical interpretation of certain holomorphic disc counts in hyperkahler manifolds given as torus fibrations [GMN10]. These disc counts are used to construct hyperkahler metrics. I hope to find the tropical interpretation by embedding these disc counts into a bigger tropical theory in which they are the terms in a wall-crossing formula. Perhaps such a theory will have enough structure to force tropical and classical counts to agree. Part of this project is developing a combinatorial interpretation of the wall-crossing formula of Kontsevich-Soibelman [KS08]. I believe that one can systematically understand it in terms of the recursive structure of combinatorial moduli spaces of tropical trees following ideas of Gross [G10].

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